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Vibration analysis of a bi-directional functionally graded nano beams with various boundary conditions using Ritz method

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Abstract
Vibration analysis of functionally graded nano beams with different boundary conditions is presented in this paper. The first-order shear deformation theory and nonlocal elasticity theory are used to incorporate size small effect of functionally graded nano beams. Ritz-type analytical solution is used to solve the characteristic equations of motion for different boundary conditions. Numerical results are presented to compare with those from earlier works, and to investigate the effects of span-to-height ratio, material parameter and scale factor on the natural frequencies of functionally graded nano beams.

Keywords: Vibration, Bi-directional functionally graded nano beams, Nonlocal elasticity theory, Various boundary conditions.

1. Introduction

Potential application of FG nano beams in recent years led to the development of the field of computational nano mechanics. In practice, the classical continuum theories fail to accurately predict the mechanical behaviour of nanostructures due to small dimensions of a such structure. To overcome this adverse, Eringen [1, 2] proposed size-dependent continuum theory known as the nonlocal elasticity theory. According to this approach, the stress at a reference point in an elastic continuum not only depends on the strain at the point but also on strains at every point of the body. Based on the nonlocal elasticity of Eringen, many researches on static, buckling and vibration of isotropic nano beams have been investigated, only some representative references are cited. Reddy [3] reformulated local beam theory by using the nonlocal differential constitutive relations of Eringen to study bending, vibration, and buckling behaviors of nano beams in which an analytical solution has been obtained to bring out the effect of the nonlocal behavior of nano beams. Aydogdu [4] proposed a generalized nonlocal beam theory to study bending, buckling, and free vibration of nano beams by using the nonlocal constitutive equations of Eringen. Xia et al. [5] used the differential quadrature method to study bending, post buckling, and free vibration for nonlinear micro beams in which a nonlinear model has been conducted within the context of non-classical continuum mechanics by introducing a material length-scale parameter. Pradhan and Murmu [6] developed a single nonlocal beam model to investigate the bending and vibration characteristics of a nano cantilever beam. Phadikar and Pradhan [7] presented finite element formulations for nonlocal elastic Euler–Bernoulli beam and Kirchhoff plate. Finite element results for bending, vibration, and buckling for nonlocal beam with four classical boundary conditions have been computed. Thai [8] proposed a nonlocal beam theory for bending, buckling and vibration of simply-supported isotropic nano beams using Navier solution. Thai and Vo [9] developed a nonlocal sinusoidal shear deformation beam theory for bending, buckling and free vibration of simply-supported nano beams. For FG nano beams, the studies on static, buckling and vibration behaviours of FG nano beams have been considered by many authors. Eltaher et al. [10] presented free vibration analysis of functionally graded (FG) size-dependent nano beams using finite element method in which the size-dependent FG nano beam has been investigated on the basis of the nonlocal continuum model. Results from this work showed the significance of the material distribution profile, nonlocal effect, and boundary conditions on the dynamic characteristics of nano beams. Ebrahimi and Salari [11] analyzed thermo-mechanical effects on vibration of nonlocal temperature dependent FG nano beams with various boundary conditions in which nonlocal Euler-Bernoulli beam theory has been used. Eltaher et al. [12] presented static and buckling responses of FG nano beams with different boundary conditions using finite element method. Ebrahimi and Salari [13] used nonlocal Timoshenko beam theory for analysis of thermal buckling and free vibration of FG nano beams in which Navier solution has been applied for analysis of simply-supported FG nano beams. Based on Timoshenko's theory, Simsek and Yurtcu [14] analyzed bending and buckling of simply supported FG nano beams using Navier solution. Ebrahimi and Barati [15] investigated effects of moisture and temperature on free vibration characteristics of simply supported FG nano beams resting on elastic foundation by developing various refined beam theories. This topic has also expanded to FG nano beams with various boundary conditions by Ebrahimi and Barati [16] using differential transform method. A literature review shows that the studies on behaviours of FG nano beams considered effects of transverse shear deformation by using nonlocal first-order shear deformation beam theory (FOBT) and nonlocal higher-order shear deformation beam theory (HOBT), and most of them studied FG simply-supported nano beams using Navier-type solution. Some of researches tried to solve FG nano beams with different boundary condition using finite element method and trigonometric series solution. Moreover, it also reveals that the number of researches considered effects of normal strain on behaviours of FG nano beams are limited. Tounsi et al. [17] proposed a nonlocal beam theory for analysis of stretching effect of isotropic nano beams. Ebrahimi and Barati [18] applied a nonlocal strain gradient elasticity theory to wave dispersion behavior of a size-dependent FG nano beam in thermal environment in which the theory contains two scale parameters corresponding to both nonlocal and strain gradient effects and a quasi-3D sinusoidal beam theory considering shear and normal deformations is employed. A literature review on the behavior analysis of FG nano beams shows that most of previous works study FG nano beams with simple-supported boundary conditions, a number of researches investigated various boundary conditions are still limited.

The objectives of this paper is to propose vibration analysis of FG nano beams with various boundary conditions. The theory is based on Timoshenko's and Eringer's nonlocal elasticity ones. Ritz solution method is used to solve characteristic equations of motion with various boundary conditions. Numerical results are compared to the earlier works and to investigate the effects of material distribution through the beam thickness, span-to-height ratio, scale length parameter and boundary conditions on the natural frequencies of FG nano beams.

2. Theoretical formulation

Consider a FG nano beam as in Fig. 1 with rectangular section $b \times h$ and length L. It is made of a mixture of isotropic ceramic and metal whose properties vary continuously in the beam thickness as follows:

$$P(x,z) = e^{p_x \alpha(x)} P(z) \quad (1)$$

with

$$\alpha(x) = \frac{x}{L}, \quad P(z) = P_{cm}\left(\frac{z}{h} + \frac{1}{2}\right)^{p_z} + P_m$$

where *P(x.z)* is material elastic moduli as Young modulus $E(x,z)$, Poisson's ratio $\nu(x,z)$, mass density $\rho(x,z)$ at location z; $P_{cm} = P_c - P_m$ are material elastic properties of ceramic and metal, respectively; p_x, p_z is power-law material parameter.

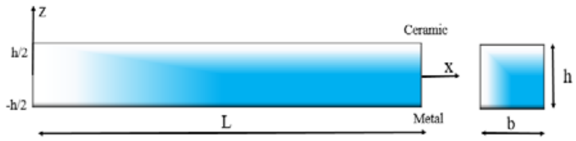


Fig. 1. Geometry of a functionally graded nano beams.

2.1. Kinetic and strain

The displacement field of FG nano beams is given by:

$$\begin{aligned} u_1(x,z,t) &= u(x,t) + z\theta(x,t) \\ u_3(x,z,t) &= w(x,t) \end{aligned} \quad (2)$$

where the comma indicates the partial differentiation with respect to the coordinate subscript that follows; $u(x,t)$, $\theta(x,t)$, $w(x,t)$ are axial displacement, rotation and transverse displacement at the mid-plan of the nano beam, respectively.

The strain field of nano beams is given by:

$$\begin{aligned} \varepsilon_x &= u_{,x} + z\theta_{,x} = \varepsilon_x^{(0)} + z\varepsilon_x^{(1)} \\ \gamma_{xz} &= \gamma_{xz}^{(0)} = \theta + w_{,x} \end{aligned} \quad (3)$$

2.2. Equations of motion

Lagrangian functional is used to derive the equations of motion:

$$\Pi = U - K \quad (4)$$

where U and K denote the strain energy and kinetic energy, respectively.

The variation of strain energy U of system is given by:

$$\begin{aligned} U &= \int_V \left(\sigma_x \varepsilon_x + \sigma_{xz}\gamma_{xz}\right) dV \\ &= \int_0^L \left(N_x \varepsilon_x^{(0)} + M_x \varepsilon_x^{(1)} + Q\gamma_{xz}^{(0)}\right) dx \end{aligned} \quad (5)$$

where the stress resultants are defined as:

$$\begin{aligned} \left(N_x, M_x\right) &= \int_{-h/2}^{h/2} (1,z)\sigma_x b dz \\ Q &= \int_{-h/2}^{h/2} k^s \sigma_{xz} b dz \end{aligned} \quad (6)$$

where k^s is shear correction factor which is supposed to be 5/6.

The variation of kinetic energy K of system is written by:

$$K = \int_V \rho(x,z)\left(\dot{u}_1^2 + \dot{u}_3^2\right)dV = \int_0^L e^{p_x \alpha(x)}\left(I_0\dot{u}^2 + I_2\dot{\theta}^2 + 2I_1\dot{u}\dot{\theta} + I_0\dot{w}^2\right)dx \quad (7)$$

where dot-superscript denotes the differentiation with respect to the time t; ρ is the mass density of each layer, and I_0, I_1, I_2 are the inertia coefficients defined by:

$$\left(I_0, I_1, I_2\right) = \int_{-h/2}^{h/2}\left(1, z, z^2\right)\rho(z)\,b\,dz \quad (8)$$

Substituting Eqs. (5) and (7) into Eq. (4)

$$\Pi = \int_0^L \Big[N_x \varepsilon_x^{(0)} + M_x \varepsilon_x^{(1)} + Q\gamma_{xz}^{(0)} - e^{p_x\alpha(x)}\left(I_0\dot{u}^2 + I_2\dot{\theta}^2 + 2I_1\dot{u}\dot{\theta} + I_0\dot{w}^2\right)\Big]dx \quad (9)$$

Leads to the following equations of motion:

$$N_{x,x} = I_0\ddot{u} + I_1\ddot{\theta}$$
$$M_{x,x} - Q = I_1\ddot{u} + I_2\ddot{\theta} \quad (10)$$
$$Q_{,x} = I_0\ddot{w}$$

2.2. Nonlocal elasticity theory for FG nano beams

Based on the Eringen's nonlocal elasticity theory [2], nonlocal constitutive equations are expressed by:

$$\left(1 - \mu\nabla^2\right)\sigma_{ij} = t_{ij} \quad (11)$$

where ∇ denotes Laplacian operator; $\mu = (e_0 a)^2$ is parameter of scale length that considers the influences of small size on the response of nanostructures with e_0 is a constant appropriate to each material, a is an internal characteristics length (e.g., lattice parameter, granular distance) and t_{ij} are global stresses. The constitutive equations of FG nano beams are hence written under the following expressions:

$$\sigma_x - \mu\sigma_{x,xx} = E(x,z)\varepsilon_x$$
$$\sigma_{xz} - \mu\sigma_{xz,xx} = G(x,z)\gamma_{xz} \quad (12)$$

where $G(x,z) = E(x,z)/\left(2 + 2\nu(z)\right)$ is shear modulus. Substituting Eqs. (3) into Eqs. (12) and then subsequent results into the stress resultants in Eqs. (6), the following nonlocal constitutive equations of stress resultants are defined as:

$$N_x - \mu N_{x,xx} = Au_{,x} + B\theta_{,x}$$
$$M_x - \mu M_{x,xx} = Bu_{,x} + D\theta_{,x} \quad (13)$$
$$Q - \mu Q_{,xx} = A^s\left(\theta + w_{,x}\right)$$

Where A, B, D, D^s are the stiffness's of FG nano beams which are defined by:

$$\left(A, B, D\right) = \int_{-h/2}^{h/2}\left(1, z, z^2\right)E(z)\,b\,dz$$
$$A^s = \int_{-h/2}^{h/2} k^s \frac{E(z)}{2 + 2\nu(z)}\,b\,dz \quad (14)$$

Substituting Eqs. (10) into Eqs. (13) leads to the expressions of stress resultants as follows:

$$N_x = \mu\left(I_0\ddot{u}_{,x} + I_1\ddot{\theta}_{,x}\right) + Au_{,x} + B\theta_{,x}$$
$$M_x = \mu\left(I_0\ddot{w} + I_1\ddot{u}_{,x} + I_2\ddot{\theta}_{,x}\right) + Bu_{,x} + D\theta_{,x} \quad (15)$$
$$Q = \mu I_0\ddot{w}_{,x} + A^s\left(\theta + w_{,x}\right)$$

Substituting Eqs. (15) into Eq. (9) yields:

$$\Pi = \frac{1}{2}\int_0^L e^{p_x\alpha(x)}\Big[Au_{,x}^2 + 2B\theta_{,x}u_{,x} + D\theta_{,x}^2 + A^s\theta^2 + 2\theta w_{,x} + A^s w_{,x}^2 + \mu\left(I_0\ddot{u}_{,x} + I_1\ddot{\theta}_{,x}\right)u_{0,x} + \mu\left(I_0\ddot{w} + I_1\ddot{u}_{,x} + I_2\ddot{\theta}_{,x}\right)\theta_{,x} + \mu I_0\ddot{w}_{,x}\left(\theta + w_{,x}\right) - \left(I_0\dot{u}_{,x}^2 + I_2\dot{\theta}^2 + 2I_1\dot{u}\,\dot{\theta} + I_0\dot{w}^2\right)\Big]dx \quad (16)$$

2.3. Ritz-type analytical solution

Based on the Ritz method, the displacements u_0, w_0, θ are approximated in the following forms:

$$u(x,t) = \sum_{j=1}^{m}\psi_j(x)u_j e^{i\omega t}$$
$$w(x,t) = \sum_{j=1}^{m}\varphi_j(x)w_j e^{i\omega t} \quad (17)$$
$$\theta(x,t) = \sum_{j=1}^{m}\psi_j(x)\theta_j e^{i\omega t}$$

where u_j, w_j, θ_j are unknown values to be determined; $i^2=-1$; ω is natural frequency; $\psi_j(x)$ and $\varphi_j(x)$ are the shape functions which are proposed in Table 1 for simply-supported (S-S), clamped-clamped (C-C) and clamped-free (C-F) boundary conditions (BC). These shape functions satisfy the BCs given in Table 2.

Table 1. Shape functions [20].

BCs	$\frac{\psi_j(x)}{e^{-jx/L}}$	$\frac{\varphi_j(x)}{e^{-jx/L}}$
S-S	$(L-2x)$	$x(L-x)$
C-F	x	x^2
C-C	$x(L-x)$	$x^2(L-x)^2$

Table 2. Kinematic BCs of nano beams.

BCs	Position	Value
S-S	x=0	$w=0$
	x=L	$w=0$
C-F	x=0	$u=0, w=0, w_{,x}=0, \theta=0$
	x=L	
C-C	x=0	$u=0, w=0, w_{,x}=0, \theta=0$
	x=L	$u=0, w=0, w_{,x}=0, \theta=0$

Substituting Eqs. (17) into Eq. (16), the following characteristic equation is obtained:

$$\left(\begin{bmatrix} \mathbf{K}^{11} & \mathbf{0} & \mathbf{K}^{13} \\ \mathbf{0} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ {}^{T}\mathbf{K}^{13} & {}^{T}\mathbf{K}^{23} & \mathbf{K}^{33} \end{bmatrix} - \omega^2 \left(\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{M}^{13} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ {}^{T}\mathbf{M}^{13} & \mathbf{0} & \mathbf{M}^{33} \end{bmatrix} + \mu \begin{bmatrix} \mathbf{M}_m^{11} & \mathbf{0} & \mathbf{M}_m^{13} \\ \mathbf{0} & \mathbf{M}_m^{22} & \mathbf{M}_m^{23} \\ {}^{T}\mathbf{M}_m^{13} & {}^{T}\mathbf{M}_m^{23} & \mathbf{M}_m^{33} \end{bmatrix}\right)\right) \begin{Bmatrix} \mathbf{u} \\ \mathbf{w} \\ \mathbf{\theta} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad (18)$$

where the components of stiffness matrix **K** and mass matrix **M** are given by:

$$K_{ij}^{11} = A\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx, K_{ij}^{13} = B\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx,$$
$$K_{ij}^{22} = A^s\int_0^L e^{p_x\alpha(x)}\varphi_{i,x}\varphi_{j,x}dx, K_{ij}^{23} = A^s\int_0^L e^{p_x\alpha(x)}\varphi_{i,x}\psi_{j}dx,$$
$$K_{ij}^{33} = D\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx + A^s\int_0^L e^{p_x\alpha(x)}\psi_{i}\psi_{j}dx \quad (19)$$

$$M_{ij}^{11} = I_0\int_0^L e^{p_x\alpha(x)}\psi_{i}\psi_{j}dx, M_{ij}^{13} = I_1\int_0^L e^{p_x\alpha(x)}\psi_{i}\psi_{j}dx,$$
$$M_{ij}^{22} = I_0\int_0^L e^{p_x\alpha(x)}\varphi_{i}\varphi_{j}dx, M_{ij}^{33} = I_2\int_0^L e^{p_x\alpha(x)}\psi_{i}\psi_{j}dx$$
$$M_{mij}^{11} = I_0\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx, M_{mij}^{13} = I_1\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx$$
$$M_{mij}^{22} = I_0\int_0^L e^{p_x\alpha(x)}\varphi_{i,x}\varphi_{j,x}dx, M_{mij}^{23} = I_0\int_0^L e^{p_x\alpha(x)}\psi_{i}\varphi_{j,x}dx,$$
$$M_{mij}^{33} = I_2\int_0^L e^{p_x\alpha(x)}\psi_{i,x}\psi_{j,x}dx \quad (20)$$

Finally, the vibration responses of FG nano beams can be determined by solving Eq. (18).

3. Numerical examples and discussions

The fundamental natural frequencies with respect to the series number p_z and p_x=0 for different boundary conditions are given in Table 3. It is observed that the responses converge quickly for three boundary conditions: p_z = 12 with L/h=10 for vibration. Thus, these numbers of series terms will be used for vibration analysis, respectively throughout the numerical examples.

Table 3. Convergence studies for fundamental frequencies of FG nano beams (p_z=1, p_x=0, μ=1(nm)^2)

L/h	BC	Numbers of series N				
		8	10	12	14	16
10	S-S	6.5836	6.5836	6.5836	6.5836	6.5836
	C-F	2.4193	2.4191	2.4190	2.4190	2.4190
	C-C	14.2664	14.2607	14.2569	14.2542	14.2522

Fig. 2 illustrate the fundamental frequencies with changing of the nonlocality parameter, material distribution at L/h=100 and the variation of boundary conditions. It can be concluded that, the frequency decreases with high rate where the power exponent in range from 0 to 4 than that the power exponent in interval between 4 and 10. The frequency decreases as the nonlocality parameter increased from 0 to 5.0*10^{-12} with the same rate.

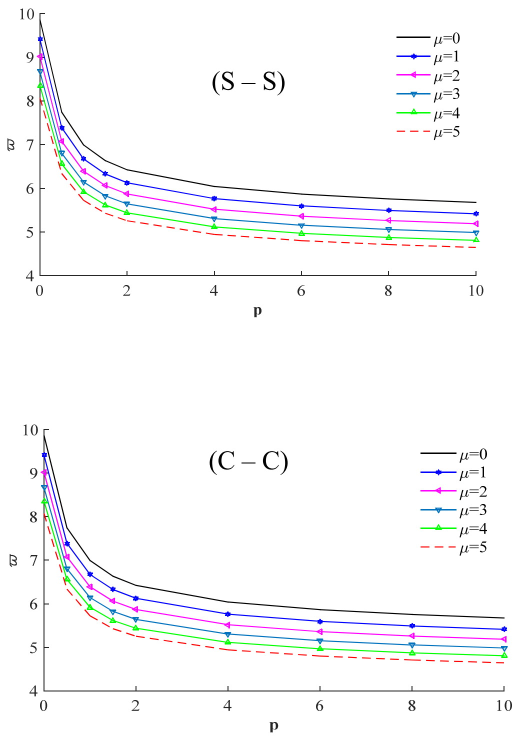
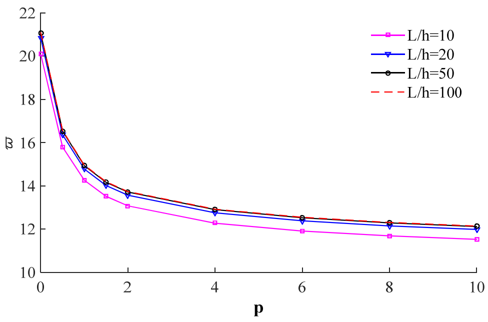


Fig. 2. The variation of the frequency with material graduation for different nonlocality parameter

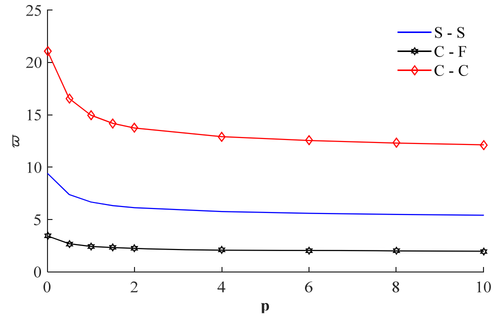
Fig 3 illustrates the frequency with the different slenderness ratio (L/h=10, 20, 50, 100). When the ratio of length and height is 10 then there is a difference than the other ratios. The ratio of length and height of 100 is almost identical to that of length and height of 50. In this study, we use the ratio L/h = 100 to study for another problem.



(C – C, μ = 1(nm)^2)

Fig. 3. The variation of the frequency with material graduation for the different span-to-height ratio

Fig 4 illustrates the nondimensional frequency with the different boundary conditions at material at graduation p=1, the nonlocality parameter μ=1(nm)^2 and the constant slenderness ratio (L/h=100). In this figure, it indicates that the dimensionless frequency will be gradually decreasing from C-C, S-S and C-F. The largest nondimensional frequency decreases when the material constant is between 0 and 2. Then, the nondimensional frequency tends to move vertically as the material constant increases.



(μ = 1(nm)^2)

Fig. 4. The variation of the frequency with material graduation for the different boundary conditions.

Tables 4-6 figure out the effects of nonlocal parameter and material graduation on the frequencies of the simply supported (S-S), clamped – free (C-F) and clamped – clamped (C-C) beams, respectively. It is concluded that, as the material graduation increases the frequencies decreased. Also, as the nonlocal parameter increases the frequencies decreased, for the both cases.

In Tables 7-9 the effect of the power-law indexes (p_x, p_z) on the nondimensional natural frequencies is illustrated for the nonlocality parameter μ=1(nm)^2 and for the slenderness ratio (L/h=100) of the simply supported (S-S), clamped – free (C-F) and clamped – clamped (C-C) beams, respectively. It is concluded that, as the material graduation increases (p_x, p_z) the frequencies decreased with S-S and C-F boundary conditions but opposite results for C-C boundary conditions. These results are also clearly shown in Figure 5.

Table 4. Variation of the nondimensional first natural frequencies with respect to the material distribution and the span-to-height ratio of S-S FG nano beams.

μ (nm)^2	L/h	Theory	Material parameter p_z (p_x=0) 0	0.5	1	5
μ=0	20	Present	9.8281	7.7141	6.9670	5.9169
		Eltaher [10]	9.8797	7.8061	7.0904	6.0025
	50	Present	9.8629	7.7412	6.9916	5.9389
		Eltaher [10]	9.8724	7.7998	7.0852	5.9990
μ =1	20	Present	9.3831	7.3647	6.6515	5.6492
		Eltaher [10]	9.4238	7.4458	6.7631	5.7256
	50	Present	9.4106	7.3862	6.6710	5.6666
		Eltaher [10]	9.4172	7.4403	6.7583	5.7218
μ =2	20	Present	8.9932	7.0587	6.3751	5.4146
		Eltaher [10]	9.0257	7.1312	6.4774	5.4837
	50	Present	9.0153	7.0759	6.3907	5.4285
		Eltaher [10]	9.0205	7.1269	6.4737	5.4808

Table 5. Variation of the nondimensional first three natural frequencies of C-F FG nano beams (L/h=100, N=10)

μ (nm)^2	Theory	Material parameter p_z (p_x=0)				
		0	0.5	1	5	10
0	Present	3.5161	2.7597	2.4925	2.1173	2.0225
	Eltaher [10]	3.5167	2.7600	2.4932	2.1168	2.0221
1	Present	3.5153	2.7606	2.4919	2.1168	2.0220
	Eltaher [10]	3.5292	2.7693	2.5134	2.1268	2.0310
2	Present	3.5145	2.7584	2.4914	2.1163	2.0215
	Eltaher [10]	3.5461	2.7841	2.5149	2.1360	2.0405
3	Present	3.5137	2.7578	2.4908	2.1158	2.0211
	Eltaher [10]	3.5632	2.8019	2.5259	2.1458	2.0498

Table 6. Variation of the nondimensional first three natural frequencies of C-C FG nano beams (L/h=100, N=10).

μ (nm)^2	Theory	Material parameter p_z (p_x=0)				
		0	0.5	1	5	10
0	Present	22.3597	17.5498	15.8506	13.4636	12.8607
	Eltaher [10]	22.3744	17.5613	15.8612	13.4733	12.8698
1	Present	21.0991	16.5604	14.9570	12.7047	12.1358
	Eltaher [10]	21.1096	16.5686	14.9645	12.7116	12.1423
2	Present	20.0255	15.7177	14.1958	12.0583	11.5183
	Eltaher [10]	20.0330	15.7235	14.2013	12.0633	11.5230
3	Present	19.0974	14.9892	13.5379	11.4995	10.9845
	Eltaher [10]	19.1028	14.9934	13.5419	11.5032	10.9880

Table 7. Variation of the nondimensional natural frequencies of S-S FG nano beams (L/h=100, N=10, μ=1(nm)^2).

Theory	Material parameter	p_z				
		0	0.5	1	5	10
Present	p_x=0	9.4146	7.3892	6.6738	5.6691	5.4152
Eltaher [10]	p_x=0	9.4162	7.4396	6.7577	5.7212	5.4384
Present	p_x=0.5	9.3894	7.3735	6.6560	5.6539	5.4008
Present	p_x=1	9.3144	7.3146	6.6028	5.6088	5.3576
Present	p_x=5	7.1758	5.6351	5.0868	4.3210	4.1275
Present	p_x=10	3.1093	2.4417	2.2041	1.8723	1.7885

Table 8. Variation of the nondimensional natural frequencies of C-C FG nano beams (L/h=100, N=10, μ=1(nm)^2).

Theory	Material parameter	p_z				
		0	0.5	1	5	10
Present	p_x=0	21.0991	16.5604	14.9570	12.7047	12.1358
Eltaher [10]	p_x=0	21.1096	16.5686	14.9645	12.7116	12.1423
Present	p_x=0.5	21.2197	16.6640	15.0424	12.7773	12.2051
Present	p_x=1	21.5626	16.9332	15.2855	12.9838	12.4023
Present	p_x=5	29.7022	23.3253	21.0555	17.8851	17.0841
Present	p_x=10	48.6493	38.2046	34.4869	29.2940	27.9821

Table 9. Variation of the nondimensional natural frequencies of C-F FG nano beams (L/h=100, N=10, μ=1(nm)^2).

Theory	Material parameter	p_z				
		0	0.5	1	5	10
Present	p_x=0	3.5153	2.7606	2.4919	2.1168	2.0220
Eltaher [10]	p_x=0	3.5292	2.7693	2.5134	2.1268	2.0310
Present	p_x=0.5	3.0097	2.3635	2.1335	1.8123	1.7312
Present	p_x=1	2.5658	2.0149	1.8189	1.5451	1.4759
Present	p_x=5	0.6277	0.4930	0.4450	0.3780	0.3611
Present	p_x=10	0.0885	0.0695	0.0627	0.0533	0.0509

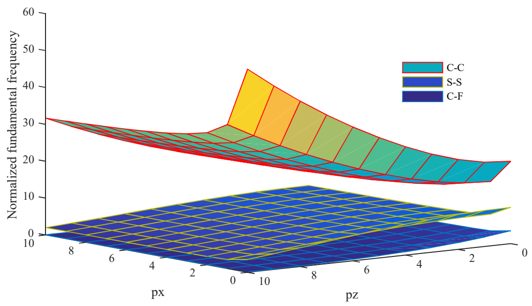


Fig. 5. The variation of the frequency with material graduation for the different boundary conditions (p_x and p_z, μ = 1(nm)^2)

4. Conclusions

The free vibration analysis of bi-directional functionally graded nano beam modeled according to Timoshenko beam theory is presented. The size-dependent (nonlocal) effect is introduced according to Eringen's nonlocal elasticity model. The variational problem governing the axial and lateral deformations is derived using the virtual-work principle. Ritz method is used to approximate the axial and lateral displacements, respectively. The fundamental frequencies of a nano beam are investigated versus the nonlocal and material-distribution parameters for different BCs of nano beam. The obtained results show that, the material-distribution profile may be manipulated to select a specific design frequency. It is also shown that, the nonlocal parameter has a notable effect on the fundamental frequencies of nano beam.

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